Report No. NADC-91005-60

AD-A232 353

WORST-CASE DESIGN OF OPTIMAL CONTROLLERS OVER A FINITE HORIZON

M. Bala Subrahmanyam
Air Vehicle And Crew Systems Technology Department
NAVAL AIR DEVELOPMENT CENTER
Warminster, PA 18974-5000

1991 JANUARY

FINAL REPORT
Period Covering September 1990 to December 1990
Task No. R36240000A
Work Unit No. 101598
Program Element No. 0602936N

Approved for Public Release; Distribution is Unlimited

Prepared for
OFFICE OF NAVAL TECHNOLOGY
Arlington, VA 22217-5000

NOTICES

REPORT NUMBERING SYSTEM — The numbering of technical project reports issued by the Naval Air Development Center is arranged for specific identification purposes. Each number consists of the Center acronym, the calendar year in which the number was assigned, the sequence number of the report within the specific calendar year, and the official 2-digit correspondence code of the Command Officer or the Functional Department responsible for the report. For example: Report No. NADC-88020-60 indicates the twentieth Center report for the year 1988 and prepared by the Air Vehicle and Crew Systems Technology Deartment. The numerical codes are as follows:

CODE	OFFICE OR DEPARTMENT
00	Commander, Naval Air Development Center
01	Technical Director, Naval Air Development Center
05	Computer Department
10	AntiSubmarine Warfare Systems Department
20	Tactical Air Systems Department
30	Warfare Systems Analysis Department
40	Communication Navigation Technology Department
50	Mission Avionics Technology Department
60	Air Vehicle & Crew Systems Technology Department
70	Systems & Software Technology Department
80	Engineering Support Group
90	Test & Evaluation Group

PRODUCT ENDORSEMENT — The discussion or instructions concerning commercial products herein do not constitute an endorsement by the Government nor do they convey or imply the license or right to use such products.

Reviewed By: [illegible] Date: 25 Jan 1991
Branch Head

Reviewed By: [illegible] Date: 2/6/91
Division Head

Approved By: [illegible] Date: 2/[illegible]/91
Director/Deputy Director

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION Unclassified	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION AVAILABILITY OF REPORT Approved for Public Release; Distribution is Unlimited
2b DECLASSIFICATION DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NADC-91005-60	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION Air Vehicle and Crew Systems Technology Department 6b OFFICE SYMBOL (If applicable) 6012	7a NAME OF MONITORING ORGANIZATION
6c ADDRESS (City, State, and ZIP Code) NAVAL AIR DEVELOPMENT CENTER Warminster, PA 18974-5000	7b ADDRESS (City, State, and ZIP Code)
8a NAME OF FUNDING/SPONSORING ORGANIZATION OFFICE OF NAVAL TECHNOLOGY 8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c ADDRESS (City, State, and ZIP Code) Arlington, VA 22217-5000	10 SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT ACCESSION NO
0602936N	0	R36240000A	101598

11 TITLE (Include Security Classification)
Worst-Case Design of Optimal Controllers over a Finite Horizon

12 PERSONAL AUTHOR(S)
M. Bala Subrahmanyam

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM 9/90 TO 12/90	1991 January	32

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Worst-Case Optimal Control, H_∞ Control, Minimax Problem. Performance Robustness, Riccati equations
01	04		

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

A worst-case optimal control problem is posed and solved making use of a minimax formulation. The emphasis is on the synthesis of optimal controllers whereas the usual H_∞ methods give conditions for the synthesis of suboptimal ones. The solution for the nonzero initial condition case is expressed as a state feedback controller. Two dynamic Riccati equations need to be solved to synthesize the optimal controller in this case. Also formulas giving the variation in performance in terms of parameter variations are derived.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a NAME OF RESPONSIBLE INDIVIDUAL M. Bala Subrahmanyam	22b TELEPHONE (Include Area Code) (215) 441-7151	22c OFFICE SYMBOL Code 6012

DD Form 1473, JUN 86 Previous editions are obsolete SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603 Unclassified

CONTENTS

CHAPTER I

Introduction

In this report, we consider an H_∞-like problem on a finite horizon. In Chapter 2 a problem with a restricted performance index is treated. We propose a worst-case optimal controller, whereas the usual H_∞ solutions yield suboptimal ones. The system considered is linear time-varying and the expressions for the worst-case exogenous input and the optimal controller are in terms of solutions of two dynamic Riccati equations in the case where the initial state is nonzero. Also an expression for the optimal controller is obtained in this case in terms of full state feedback. Since the procedure to obtain the optimal controller is a noniterative one, the computational time is greatly reduced. Also, a novel feature is the derivation of a formula for the performance variation of the optimal controller in terms of variations in the system matrices.

In Chapter 3 we consider a problem with a generalized performance index. We develop necessary conditions for a minimax problem involving control and exogenous inputs. Again the problem can be regarded as a finite horizon version of the H_∞ optimal control problem. The emphasis is on the synthesis of optimal controllers whereas the usual H_∞ methods give conditions for the synthesis of suboptimal ones. Feedback controllers are developed for the case of nonzero initial conditions. Also, expressions are derived for the variation in performance in terms of system parameter variations. These linear expressions are useful in the evaluation of the robustness of the proposed optimal control strategy.

CHAPTER II

Worst-Case Optimal Control with a Restricted Performance Criterion

1. INTRODUCTION

There are several recent papers attacking the H_∞ problem from a state space point of view and all of these obtain a characterization of suboptimal output-feedback controllers[1-5]. The suboptimal controller is usually obtained by solving two Riccati equations. There are also finite interval versions of these solutions and extensions have been made to the linear time-varying case as well[4,5]. The state space approach has yielded new insights into the features of the H_∞ controller, and one of these is the separation of the control problem into a full state feedback design and an observer design.

In a different approach taken by this author[6-8], a measure of peformance is computed for a given controller and nonlinear programming algorithms are utilized to find a controller that optimizes the performance. This approach is suitable for extending the methodology to solve problems involving convex functionals[9]. We have also applied the methodology to solve model reduction problems[10]. One of the main advantages of this approach is the quantification of variation in peformance when uncertainties are present in the system matrices. However, it is tedious to compute the optimal controller in this case because it requires several iterations.

The main contribution of this report is the noniterative characterization of the optimal controller. We consider a restricted performance criterion in Chapter 2 and a generalized performance criterion in Chapter 3. The full state feedback solution in the nonzero initial state case is in terms of two dynamic Riccati equations. The integration of these equations is easy since only one of the Riccati equations depends on the solution of the other. Unlike the usual approaches which yield suboptimal controllers, our approach yields an optimal controller. Thus the need for iterative solution methods like the γ-iteration is eliminated. Because of the dynamic Riccati equations, the control gain matrix will be time-varying even when the system matrices are time-invariant. An important by-product of the approach is a formula for the variation of performance in terms of variations in the system matrices. These variations in performance are useful in evaluating the robustness of the proposed controller.

2. PROBLEM FORMULATION

The linear time-varying system is given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \qquad x(t_0) = x_0, \tag{1}$$

$$z = C(t)x + D(t)u, \tag{2}$$

where x, u, v, and z represent the state vector, the control vector, the exogenous input vector, and the vector to be controlled respectively. We consider the minimax problem

$$\min_v \max_u \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^T \frac{1}{2}v^*(t)R(t)v(t)\,dt}{\int_{t_0}^T \frac{1}{2}z^*(t)W(t)z(t)\,dt}, \tag{3}$$

where $R(t)$ and $W(t)$ are positive definite matrices and the superscript * denotes matrix or vector transpose. Also S_1 is a constant positive definite matrix. The above problem is related to the H_∞ problem since the functional in (3) represents the ratio of exogenous signal energy to the error energy. Also, the solution procedure given in the following sections is extended in Chapter III to the case where (2) is of the form $z = C(t)x + D(t)u + E(t)v$.

3. OPTIMAL SOLUTIONS

Let

$$J(u, v) = \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^T \frac{1}{2}v^*(t)R(t)v(t)\,dt}{\int_{t_0}^T \frac{1}{2}z^*(t)W(t)z(t)\,dt}. \tag{4}$$

Using (2), we can write (4) as

$$J(u,v) = \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^*(t) R(t) v(t)\, dt}{\int_{t_0}^{T} \{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u\}\, dt}. \tag{5}$$

Notice that the weighting matrices W_1, W_2, and W_3 are time-varying.

We will first of all maximize (4) over u for any given $v(t) \not\equiv 0$. Thus we need to minimize

$$\int_{t_0}^{T} \{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u\}\, dt \tag{6}$$

over u assuming that $v(t)$ is given. From the maximum principle[11], which in this case is also a sufficient condition for optimality, the Hamiltonian is given by

$$H = -\{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u\} + \psi^* \{A(t)x + B_1(t)u + B_2(t)v\}, \tag{7}$$

where the adjoint variable ψ satisfies

$$\frac{d\psi}{dt} = -\frac{\partial H}{\partial x} = W_1 x + W_2 u - A^* \psi, \tag{8}$$

with

$$x(t_0) = x_0, \ \ \psi(T) = 0. \tag{9}$$

Also, setting $\frac{\partial H}{\partial u} = 0$ and assuming that W_3 is invertible for all $t \in [t_0, T]$,

$$u = W_3^{-1}(B_1^* \psi - W_2^* x). \tag{10}$$

Let

$$\begin{aligned} \hat{A} &= A - B_1 W_3^{-1} W_2^*, \\ \hat{B} &= B_1 W_3^{-1} B_1^*, \\ \hat{C} &= W_1 - W_2 W_3^{-1} W_2^*. \end{aligned} \tag{11}$$

Thus we have

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} + \begin{pmatrix} B_2 \\ 0 \end{pmatrix} v, \tag{12}$$

with

$$x(t_0) = x_0, \ \psi(T) = 0. \tag{13}$$

Let

$$\zeta = \begin{pmatrix} x \\ \psi \end{pmatrix}, \tag{14}$$

$$M = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix}, \tag{15}$$

and

$$N = \begin{pmatrix} B_2 \\ 0 \end{pmatrix}. \tag{16}$$

By (10) the denominator of (5) can be put in the form $\frac{1}{2}\zeta^* Q \zeta$, where $Q(t)$ is symmetric and positive semi-definite. The system given by (12) can be written as

$$\dot{\zeta} = M(t)\zeta + N(t)v, \tag{17}$$

with

$$x(t_0) = x_0, \ \psi(T) = 0, \tag{18}$$

and v needs to be selected to minimize the cost

$$\frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^*(t) R(t) v(t)\, dt}{\int_{t_0}^{T} \frac{1}{2}\zeta^*(t) Q(t) \zeta(t)\, dt}. \tag{19}$$

We now state the conditions that are satisfied by an optimal $v(t)$.

THEOREM 1. *Consider the system given by* (17)-(19). *If* $v_0(t)$ *minimizes* (19), *then there exists a nonzero* $\eta(t) = (\, p^*(t) \quad q^*(t)\,)^*$ *such that*

$$\frac{d\eta}{dt} = -M^*\eta - \lambda Q \zeta, \tag{20}$$

where $p(t)$ *and* $q(t)$ *are components of the adjoint vector corresponding to* $x(t)$ *and* $\psi(t)$ *respectively, such that*

$$\begin{gathered} x(t_0) = x_0, \ \psi(T) = 0, \\ p(T) = 0, \ q(t_0) = 0, \end{gathered} \tag{21}$$

where

$$\lambda = \inf_{v} \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^* R v \, dt}{\int_{t_0}^{T} \frac{1}{2}\zeta^* Q \zeta \, dt}, \tag{22}$$

and

$$v_0(t) = R^{-1} N^* \eta. \tag{23}$$

If in addition $x_0 \neq 0, p(t_0) = S_1 x(t_0)$.

Proof. If $v_0(t)$ minimizes (19), then it also minimizes

$$\tilde{J}(v) \triangleq \frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^* R v \, dt - \lambda \int_{t_0}^{T} \frac{1}{2}\zeta^* Q \zeta \, dt. \tag{24}$$

By the maximum principle[11], there exists an adjoint response $\eta(t)$ such that the Hamiltonian

$$H(\eta, \zeta, v) = -\frac{1}{2} v^* R v + \frac{1}{2}\lambda \zeta^* Q \zeta + \eta^* \{ M\zeta + Nv \} \tag{25}$$

is maximized almost everywhere on $[t_0, T]$ by $v_0(t)$. Satisfaction of $\frac{\partial H}{\partial v} = 0$ yields

$$v_0(t) = R^{-1} N^* \eta. \tag{26}$$

The adjoint variable η satisfies

$$\frac{d\eta}{dt} = -\frac{\partial H}{\partial \zeta} = -M^* \eta - \lambda Q \zeta. \tag{27}$$

By the transversality conditions, we get the boundary conditions. $\square$

Thus we have a two point boundary value problem given by

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} M & NR^{-1}N^* \\ -\lambda Q & -M^* \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}, \tag{28}$$

with

$$\begin{gathered} x(t_0) = x_0, \;\; \psi(T) = 0, \\ p(T) = 0, \;\; q(t_0) = 0, \\ p(t_0) = S_1 x(t_0) \text{ if } x_0 \neq 0. \end{gathered} \tag{29}$$

We now give a criterion for the estimation of λ. Notice that $\lambda = \min_v \max_u J(u, v)$ and gives a measure of performance of the optimal controller under worst-case conditions corresponding to $v_0(t)$. In the H_∞ case, the evaluation of λ would entail the γ-iteration.

THEOREM 2. *Let λ be the smallest positive value for which the boundary value problem given by* (28) *and* (29) *has a solution* (ζ, η) *with* $\int_{t_0}^{T} \frac{1}{2}\zeta^* Q\zeta \, dt > 0$. *Then λ is the minimum value of* (19), (ζ, η) *is an optimal pair and* $v \triangleq R^{-1}N^*\eta$ *is the worst exogenous input.*

Proof. It is clear from Theorem 1 that if $v_0(t)$ minimizes (19), then it satisfies (28) and (29), with λ being the minimum value of (19). Now suppose (ζ, η) satisfies (28) and (29) for some λ. Let $v = R^{-1}N^*\eta$.

We have

$$\begin{aligned} x_0^* S_1 x_0 + \int_{t_0}^{T} v^* R v \, dt &= x_0^* p(t_0) + \int_{t_0}^{T} (R^{-1}N^*\eta, N^*\eta) \, dt \\ &= x_0^* p(t_0) + \int_{t_0}^{T} (NR^{-1}N^*\eta, \eta) \, dt \\ &= x_0^* p(t_0) + \int_{t_0}^{T} (\dot{\zeta}, \eta) \, dt - \int_{t_0}^{T} (M\zeta, \eta) \, dt. \end{aligned} \tag{30}$$

Integrating the first integral in (30) by parts and making use of (29), we get

$$x_0^* S_1 x_0 + \int_{t_0}^{T} v^* R v \, dt = \lambda \int_{t_0}^{T} \zeta^* Q \zeta \, dt. \tag{31}$$

Thus, the cost associated with v is λ. Hence, if (ζ, η) is a solution of the boundary value problem given by (28) and (29) for the smallest parameter $\lambda > 0$, then λ is the optimal value and (ζ, η) is an optimal pair. □

Note that the boundary value problem (28)-(29) has a solution with a nonvanishing denominator for (19) for at most a countably infinite values of λ. Theorem 2 gives a sufficient condition for an exogenous input to be optimal. Thus, Theorems 1 and 2 give a complete characterization of the worst-case exogenous input.

4. COMPUTATION OF λ

In this section, we consider the boundary value problem given by (28) and (29) assuming that $x(t_0) \neq 0$. Analogous theory can be developed in case $x_0 = 0$. Making use of the transition matrix, the solution of (28) can be expressed as

$$\begin{pmatrix} x(t) \\ \psi(t) \\ p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) & \phi_{13}(t,t_0) & \phi_{14}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) & \phi_{23}(t,t_0) & \phi_{24}(t,t_0) \\ \phi_{31}(t,t_0) & \phi_{32}(t,t_0) & \phi_{33}(t,t_0) & \phi_{34}(t,t_0) \\ \phi_{41}(t,t_0) & \phi_{42}(t,t_0) & \phi_{43}(t,t_0) & \phi_{44}(t,t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \\ p(t_0) \\ q(t_0) \end{pmatrix}. \tag{32}$$

The boundary conditions given by (29) yield

$$\begin{pmatrix} \phi_{21}(T,t_0) + \phi_{23}(T,t_0)S_1 & \phi_{22}(T,t_0) \\ \phi_{31}(T,t_0) + \phi_{33}(T,t_0)S_1 & \phi_{32}(T,t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \end{pmatrix} = 0. \tag{33}$$

Let

$$\tilde{\phi} = \begin{pmatrix} \phi_{21} + \phi_{23}S_1 & \phi_{22} \\ \phi_{31} + \phi_{33}S_1 & \phi_{32} \end{pmatrix}. \tag{34}$$

In view of (33) and (28)-(29), we have $\det\left(\tilde{\phi}(T,t_0)\right) = 0$ if and only if the solution (ζ, η) of (28)-(29) is not identically zero. Thus, we need the least positive λ which makes $\det\left(\tilde{\phi}(T,t_0)\right) = 0$ and the denominator of (19) positive. This can be obtained by doing a search with λ over an interval on which there is a change in the sign of the determinant.

We found the following algorithm to be numerically more stable since numbers of lesser magnitude are involved in the computation of the transition matrices in (35). We have

$$\begin{pmatrix} \zeta(T) \\ \eta(T) \end{pmatrix} = \phi(T, \frac{T+t_0}{2})\phi(\frac{T+t_0}{2}, t_0) \begin{pmatrix} \zeta(t_0) \\ \eta(t_0) \end{pmatrix}. \tag{35}$$

Let

$$\phi^{-1}(T, \frac{T+t_0}{2}) = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix}, \tag{36}$$

and

$$\phi(\frac{T+t_0}{2}, t_0) = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} & \nu_{14} \\ \nu_{21} & \nu_{22} & \nu_{23} & \nu_{24} \\ \nu_{31} & \nu_{32} & \nu_{33} & \nu_{34} \\ \nu_{41} & \nu_{42} & \nu_{43} & \nu_{44} \end{pmatrix}. \tag{37}$$

Making use of $p(t_0) = S_1 x(t_0), q(t_0) = \psi(T) = p(T) = 0$, we have

$$\begin{pmatrix} \xi_{11} & \xi_{14} \\ \xi_{21} & \xi_{24} \\ \xi_{31} & \xi_{34} \\ \xi_{41} & \xi_{44} \end{pmatrix} \begin{pmatrix} x(T) \\ q(T) \end{pmatrix} = \begin{pmatrix} \nu_{11} + \nu_{13}S_1 & \nu_{12} \\ \nu_{21} + \nu_{23}S_1 & \nu_{22} \\ \nu_{31} + \nu_{33}S_1 & \nu_{32} \\ \nu_{41} + \nu_{43}S_1 & \nu_{42} \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \end{pmatrix}. \tag{38}$$

The above equation has a nontrivial solution if and only if

$$\det\begin{pmatrix} \xi_{11} & \xi_{14} & \nu_{11}+\nu_{13}S_1 & \nu_{12} \\ \xi_{21} & \xi_{24} & \nu_{21}+\nu_{23}S_1 & \nu_{22} \\ \xi_{31} & \xi_{34} & \nu_{31}+\nu_{33}S_1 & \nu_{32} \\ \xi_{41} & \xi_{44} & \nu_{41}+\nu_{43}S_1 & \nu_{42} \end{pmatrix} = 0. \tag{39}$$

Thus, we need the least positive λ which makes the above determinant zero.

5. SOLUTION IN TERMS OF RICCATI EQUATIONS

We now give the optimal solution in terms of solutions of two dynamic Riccati equations in the case where $x_0 \neq 0$.

THEOREM 3. *Let* $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ *be the solution of the initial value problem*

$$\dot{P} + PM + M^*P + PNR^{-1}N^*P + \lambda Q = 0, \tag{40}$$

$$P(t_0) = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{41}$$

If $x(t_0) \neq 0$, *let* Z *be the solution of the dynamic Riccati equation*

$$\dot{Z} + Z(\hat{A} + B_2R^{-1}B_2^*P_{11}) + \hat{A}^*Z + Z(\hat{B} + B_2R^{-1}B_2^*P_{12})Z - \hat{C} = 0, \quad Z(T) = 0. \tag{42}$$

Then the worst-case exogenous input is given by

$$v = R^{-1}B_2^*(P_{11} + P_{12}Z)x, \tag{43}$$

and the worst-case optimal controller is

$$u = W_3^{-1}(B_1^*Z - W_2^*)x. \tag{44}$$

Proof. Letting $\eta = P\zeta$ in (28), we get (40). From (26) the worst-case exogenous input is $v = R^{-1}N^*P\zeta$. If $x(t_0) \neq 0$, letting $\psi = Zx$, we get $v = R^{-1}B_2^*(P_{11} + P_{12}Z)x$. Also, equations (12) and (13) yield (42). From (10), we get the worst-case optimal controller given by (44). □

Note that the worst-case optimal control given by (44) has time-varying feedback gain even when the linear system is time-invariant because of the presence of $Z(t)$.

6. Performance Robustness

In this section we develop a formula for the variation of λ when there are parameter variations in the system matrices. Note that this expression for the variation in λ takes into account the corresponding variations in the optimal controller and the worst-exogenous input. Usually when a controller is synthesized with respect to the nominal values of system matrices, its gains do not change with parameter variations. Hence, later on in this section we will derive an expression for the variation in λ assuming that there is no variation of the optimal gain matrix.

For this consider (1) and (2). Let μ denote the variation in λ for elemental variations $\delta A, \delta B_1, \delta B_2, \delta C$, and δD in the matrices A, B_1, B_2, C, and D. From equations (28) and (29) of Section 3, we have the following boundary value problem given by

$$\dot{\zeta} = M\zeta + NR^{-1}N^*\eta, \tag{45}$$

$$\dot{\eta} = -\lambda Q\zeta - M^*\eta, \tag{46}$$

with

$$\begin{aligned} &x(t_0) = x_0, \psi(T) = 0, \\ &p(T) = 0, q(t_0) = 0, \\ &p(t_0) = S_1 x(t_0) \text{ if } x_0 \neq 0. \end{aligned} \tag{47}$$

To simplify the derivation, let $\delta M, \delta N$, and δQ be the variations in M, N, and Q owing to the variations $\delta A, \delta B_1, \delta B_2, \delta C$, and δD We now derive an expression for μ in terms of the variations $\delta M, \delta N$, and δQ.

Let ζ_1 and η_1 represent variations in ζ and η owing to $\delta M, \delta N$, and δQ. Let the corresponding variation in λ be denoted by μ. We have the following set of equations that are satisfied by ζ_1 and η_1:

$$\dot{\zeta}_1 = M\zeta_1 + NR^{-1}N^*\eta_1 + \delta M\zeta + (\delta NR^{-1}N^* + NR^{-1}\delta N^*)\eta, \tag{48}$$

$$\dot{\eta}_1 = -\lambda Q\zeta_1 - M^*\eta_1 - (\lambda\delta Q + \mu Q)\zeta - \delta M^*\eta, \tag{49}$$

with

$$\begin{aligned} &x_1(t_0) = x_{10}, \psi_1(T) = 0, \\ &p_1(T) = 0, q_1(t_0) = 0, \\ &p_1(t_0) = S_1 x_{10} \text{ if } x_0 \neq 0. \end{aligned} \tag{50}$$

Note that the subscript 1 of a variable in (50) denotes the corresponding variation of that variable.

THEOREM 4. *The variation μ in performance is given by*

$$\mu = \frac{-\int_{t_0}^{T}\{\lambda\zeta^*\delta Q\zeta + 2\zeta^*\delta M^*\eta + \eta^*(\delta NR^{-1}N^* + NR^{-1}\delta N^*)\eta\}\,dt}{\int_{t_0}^{T}\zeta^*Q\zeta\,dt}. \tag{51}$$

Proof. From (49), we get

$$\int_{t_0}^{T}\zeta^*\dot{\eta}_1\,dt = -\int_{t_0}^{T}\{\lambda\zeta^*Q\zeta_1 + \zeta^*M^*\eta_1 + \zeta^*(\lambda\delta Q + \mu Q)\zeta + \zeta^*\delta M^*\eta\}\,dt. \tag{52}$$

Integrating the left side of (52) by parts and making use of (45) and (50), we get

$$\begin{aligned} x_0^*S_1x_{10} + \int_{t_0}^{T}\eta^*NR^{-1}N^*\eta_1\,dt = \lambda\int_{t_0}^{T}\zeta^*Q\zeta_1\,dt \\ + \int_{t_0}^{T}\zeta^*(\lambda\delta Q + \mu Q)\zeta\,dt + \int_{t_0}^{T}\zeta^*\delta M^*\eta\,dt. \end{aligned} \tag{53}$$

By (46), the first integral on the right side of (53) is written as

$$\lambda\int_{t_0}^{T}\zeta^*Q\zeta_1\,dt = -\int_{t_0}^{T}(\dot{\eta} + M^*\eta)^*\zeta_1\,dt. \tag{54}$$

An integration by parts and equations (48) and (50) yield

$$\begin{aligned} \lambda\int_{t_0}^{T}\zeta^*Q\zeta_1\,dt = x_0^*S_1x_{10} + \int_{t_0}^{T}\eta^*NR^{-1}N^*\eta_1\,dt + \int_{t_0}^{T}\eta^*\delta M\zeta\,dt \\ + \int_{t_0}^{T}\eta^*(\delta NR^{-1}N^* + NR^{-1}\delta N^*)\eta\,dt. \end{aligned} \tag{55}$$

Substituting (55) in (53) and simplifying, we get (51). □

Since μ given by (51) is linear in the elemental variations $\delta A, \delta B_1, \delta B_2, \delta C$, and δD, at least in the time-invariant case the worst degradation in performance can be easily obtained once the range of uncertainty of the parameters is known.

Now we consider the case where $x_0 \neq 0$. Assume that the state feedback controller is determined by the nominal system matrices and is fixed. We derive a formula for the variation of λ under these conditions. Since λ gives a measure of performance of the optimal controller under worst-case conditions, we can get an idea of the degradation in performance owing to parameter variations. Equation (44) is written as

$$u = W_3^{-1}(B_1^* Z - W_2^*)x = K(t)x, \tag{56}$$

where $K(t)$ is now fixed. Let $\tilde{A} = A + B_1 K$ and $\tilde{W} = (C + DK)^* W (C + DK)$. Equation (1) can be written as

$$\dot{x} = \tilde{A}(t) + B_2(t)v, \tag{57}$$

with v chosen to minimize

$$\frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^* R v \, dt}{\int_{t_0}^{T} \frac{1}{2} x^* \tilde{W} x \, dt}. \tag{58}$$

Note that

$$\lambda = \min_{v} \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^* R v \, dt}{\int_{t_0}^{T} \frac{1}{2} x^* \tilde{W} x \, dt}. \tag{59}$$

The above minimization problem yields the two-point boundary value problem

$$\begin{pmatrix} \dot{x} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} \tilde{A} & B_2 R^{-1} B_2^* \\ -\lambda \tilde{W} & -\tilde{A}^* \end{pmatrix} \begin{pmatrix} x \\ \beta \end{pmatrix}, \tag{60}$$

$$x(t_0) = 0, \beta(t_0) = S_1 x_0, \beta(T) = 0, \tag{61}$$

where β is the adjoint variable and the worst exogenous input $v = R^{-1} B_2^* \beta$. Let $\tilde{B} = B_2 R^{-1} B_2^*$.

Let $\delta\tilde{A}, \delta\tilde{B}$, and $\delta\tilde{W}$ be the variations in $\tilde{A}, \tilde{B}$, and $\tilde{W}$ corresponding to $\delta A, \delta B_1, \delta B_2, \delta C$, and δD. Note that since $K(t)$ is fixed, $\delta\tilde{A} = \delta A + \delta B_1 K$. Let the variation in λ be now denoted by $\tilde{\mu}$. Utilizing a similar analysis as in the derivation of (51), we can get

$$\tilde{\mu} = \frac{-\int_{t_0}^{T}\{\lambda x^*\delta\tilde{W}x + 2x^*\delta\tilde{A}^*\beta + \beta^*\delta\tilde{B}\beta\}\,dt}{\int_{t_0}^{T} x^*\tilde{W}x\,dt}. \tag{62}$$

Since $\tilde{\mu}$ is linear in the variations, the worst degradation in the performance of the optimal controller can be easily computed in the time-invariant case. The worst value of $\tilde{\mu}$ gives an idea of the measure of performance robustness of the optimal controller

7. An Example

In order to illustrate the basic theory, we will work out a simple example. The system is described by the equation

$$\dot{x} = -x + u + v, \qquad x(0) = x_0 \neq 0, \tag{63}$$

and the objective is to choose u and v such that

$$\min_{v}\max_{u} \frac{\frac{1}{2}x_0^2 + \int_0^1 \frac{1}{2}v^2\,dt}{\int_0^1 \frac{1}{2}(x^2+u^2)\,dt} \tag{64}$$

is attained.

First of all, minimizing $\int_0^1 \frac{1}{2}(x^2+u^2)\,dt$ over $u(t)$ for a given $v(t)$, we get

$$\begin{pmatrix}\dot{x}\\ \dot{\psi}\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ \psi\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}v, \tag{65}$$

$$u = \psi, \tag{66}$$

$$x(0) = x_0, \psi(1) = 0, \tag{67}$$

where ψ is the adjoint variable. Now we need to choose v to minimize

$$\frac{\frac{1}{2}x_0^2 + \int_0^1 \frac{1}{2}v^2\,dt}{\int_0^1 \frac{1}{2}(x^2+\psi^2)\,dt}. \tag{68}$$

Let λ be the minimum value of (68). Denoting the adjoint variables associated with x and ψ by η^1 and η^2 respectively, we get

$$\frac{d\eta^1}{dt} = -\lambda x + \eta^1 - \eta^2, \tag{69}$$

$$\frac{d\eta^2}{dt} = -\lambda\psi - \eta^1 - \eta^2, \tag{70}$$

$$\eta^1(0) = x_0, \eta^2(0) = 0, \eta^1(1) = 0, \tag{71}$$

$$v = \eta^1. \tag{72}$$

Thus, we have

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \\ \dot{\eta}^1 \\ \dot{\eta}^2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -\lambda & 0 & 1 & -1 \\ 0 & -\lambda & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ \psi \\ \eta^1 \\ \eta^2 \end{pmatrix}, \tag{73}$$

$$x(0) = x_0, \psi(1) = 0, \eta^1(0) = x_0, \eta^2(0) = 0, \eta^1(1) = 0. \tag{74}$$

According to the theory of Section 3, λ is the least positive value for which the boundary value problem (73)-(74) has a nonzero solution.

Let ϕ be the transition matrix of the system given by (73) at $t = 1$. Solving (73) and employing the boundary conditions at $t = 1$, we get

$$0 = \begin{pmatrix} \psi(1) \\ \eta^1(1) \end{pmatrix} = F(\lambda) \begin{pmatrix} x(0) \\ \psi(0) \end{pmatrix}, \tag{75}$$

where

$$F(\lambda) = \begin{pmatrix} \phi_{21} + \phi_{23} & \phi_{22} \\ \phi_{31} + \phi_{33} & \phi_{32} \end{pmatrix}. \tag{76}$$

Thus, we need the first positive λ which makes $\det\big(F(\lambda)\big) = 0$. This value of λ is 2. It can be easily shown that with the initial condition $x(0) = 0$, the value of λ would have been 6.1159. The case of $u = cx, x(0) = 0$, where c is a constant gain is solved in [7] and in this case $\lambda = 5.6837$.

Now the Riccati equations in Theorem 3 can be easily solved to obtain the worst-case optimal controller and the worst-case exogenous input.

8. CONCLUSIONS

In this chaper we presented a solution to the finite interval worst-case state feedback controller in terms of solutions of two dynamic Riccati equations. These equations are easy to solve since only one of the two equations is dependent on the solution of the other. the procedure yields optimal solutions instead of suboptimal ones normally obtained by H_∞ methods. Also, an expression is derived for the degradation in performance of the optimal controller in terms of parameter variations.

CHAPTER III

Worst-Case Optimal Control with a Generalized Performance Criterion

1. INTRODUCTION

Recent state space approaches characterize suboptimal H_∞ controllers in terms of solutions of two Riccati equations[1-5]. Although there have been extensions of the state space approach to cases involving nonzero initial conditions[3], time-varying systems[4,5], and control on a finite horizon[5], there have been virtually no attempts to characterize the optimal solutions.

In a different approach taken by this author[6-9], the controller is assumed to be in feedback form and a performance measure is evaluated for any given controller. Then nonlinear programs can be utilized to select a controller which maximizes the performance. Although this approach yielded satisfactory controllers in several practical cases, it also consumed excessive amounts of computational time. In [10] this approach is successfully employed to solve a model reduction problem.

In this chapter we consider a worst-case optimal control problem with a generalized performance criterion. We employ a new approach by considering the underlying minimax problem and treating the adjoint variables associated with the maximization problem as state variables for the minimization problem. The associated performance index is computed in terms of the least positive value for which a certain boundary value problem has a nontrivial solution. A simple criterion for the evaluation of the performance index is given in Section 4. In the H_∞ case, the evaluation of the performance index would entail the γ-iteration. Our technique is noniterative and hence is computationally efficient. Also, expressions for the optimal feedback controller for the nonzero initial condition case are developed in terms of solutions of two dynamic Riccati equations. These Riccati equations are easy to solve since only one equation depends on the solution of the other. In Section 6 expressions for the variation in performance of the optimal controller are derived in terms of variations in the system matrices. Utilizing these expressions, the degradation in the performance of the optimal controller owing to variations in the system matrices can be easily computed. The worst degradation in the performance gives an idea of the robustness of the proposed controller.

2. PROBLEM FORMULATION

The linear time-varying system is given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \qquad x(t_0) = x_0, \tag{1}$$

$$z = C(t)x + D(t)u + E(t)v, \tag{2}$$

where x, u, v, and z represent the state vector, the control vector, the exogenous input vector, and the vector to be controlled respectively. We consider the minimax problem

$$\min_{v} \max_{u} \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2}v^*(t)R(t)v(t)\,dt}{\int_{t_0}^{T} \frac{1}{2}z^*(t)W(t)z(t)\,dt}, \tag{3}$$

where $R(t)$ and $W(t)$ are positive definite matrices and the superscript * denotes matrix or vector transpose. Also S_1 is a constant positive definite matrix. The above problem is related to the H_∞ problem since the functional in (3) represents the ratio of exogenous signal energy to the error energy. Problems where $x_0 \neq 0$ have been considered in [3]. However, [3] characterizes suboptimal solutions, whereas we characterize the optimal solutions in this report.

3. OPTIMAL SOLUTIONS

Let

$$J(u,v) = \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2}v^*(t)R(t)v(t)\,dt}{\int_{t_0}^{T} \frac{1}{2}z^*(t)W(t)z(t)\,dt}. \tag{4}$$

Using (2), we can write (4) as

$$J(u,v) = \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^*(t) R(t) v(t)\, dt}{\int_{t_0}^{T} \{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2}u^* W_3 u + x^* W_4 v + \frac{1}{2} v^* W_5 v + u^* W_6 v\}\, dt}. \quad (5)$$

Notice that the weighting matrices W_1, W_2, W_3, W_4, W_5 and W_6 are time-varying.

We will first of all maximize (4) over u for any given $v(t) \not\equiv 0$. Thus we need to minimize

$$\int_{t_0}^{T} \{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2}u^* W_3 u + x^* W_4 v + \frac{1}{2} v^* W_5 v + u^* W_6 v\}\, dt \quad (6)$$

over u assuming that $v(t)$ is given. From the maximum principle[11], which in this case is also a sufficient condition for optimality, the Hamiltonian is given by

$$\begin{aligned} H = -\{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2}u^* W_3 u + x^* W_4 v + \frac{1}{2} v^* W_5 v + u^* W_6 v\} + \\ \psi^* \{A(t)x + B_1(t)u + B_2(t)v\}, \end{aligned} \quad (7)$$

where the adjoint variable ψ satisfies

$$\frac{d\psi}{dt} = -\frac{\partial H}{\partial x} = W_1 x + W_2 u + W_4 v - A^* \psi, \quad (8)$$

with

$$x(t_0) = x_0, \;\; \psi(T) = 0. \quad (9)$$

Also, setting $\frac{\partial H}{\partial u} = 0$ and assuming that W_3 is invertible for all $t \in [t_0, T]$,

$$u = W_3^{-1}(B_1^* \psi - W_2^* x - W_6 v). \quad (10)$$

Let

$$\begin{aligned} \hat{A} &= A - B_1 W_3^{-1} W_2^*, \\ \hat{B} &= B_1 W_3^{-1} B_1^*, \\ \hat{C} &= W_1 - W_2 W_3^{-1} W_2^*, \\ G_1 &= B_2 - B_1 W_3^{-1} W_6, \\ G_2 &= W_4 - W_2 W_3^{-1} W_6. \end{aligned} \quad (11)$$

Thus we have

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} v, \tag{12}$$

with

$$x(t_0) = x_0, \ \psi(T) = 0. \tag{13}$$

Let

$$\zeta = \begin{pmatrix} x \\ \psi \end{pmatrix}, \tag{14}$$

$$M = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix}, \tag{15}$$

and

$$N = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}. \tag{16}$$

By (10) the denominator of (5) can be put in the form $\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v$, where $Q_1(t)$ and $Q_3(t)$ are symmetric and positive semi-definite. The system given by (12) can be written as

$$\dot{\zeta} = M(t)\zeta + N(t)v, \tag{17}$$

with

$$x(t_0) = x_0, \ \psi(T) = 0, \tag{18}$$

and v needs to be selected to minimize the cost

$$\frac{\frac{1}{2} x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^*(t) R(t) v(t)\, dt}{\int_{t_0}^{T} \{\frac{1}{2}\zeta^*(t) Q_1(t) \zeta(t) + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v\}\, dt}. \tag{19}$$

We now state the conditions that are satisfied by an optimal $v(t)$.

THEOREM 1. *Consider the system given by* (17)-(19). *Assume that* $R - \lambda Q_3$ *is invertible for all* $t \in [t_0, T]$. *If* $v_0(t)$ *minimizes* (19), *then there exists a nonzero* $\eta(t) = (\, p^*(t) \quad q^*(t)\,)^*$ *such that*

$$\frac{d\eta}{dt} = -M^*\eta - \lambda Q_1 \zeta - \lambda Q_2 v, \tag{20}$$

where $p(t)$ and $q(t)$ are components of the adjoint vector corresponding to $x(t)$ and $\psi(t)$ respectively, such that

$$x(t_0) = x_0, \ \psi(T) = 0,$$
$$p(T) = 0, \ q(t_0) = 0, \tag{21}$$

where

$$\lambda = \inf_v \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^T \frac{1}{2}v^* R v \, dt}{\int_{t_0}^T \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2}v^* Q_3 v\} \, dt}, \tag{22}$$

and

$$v_0(t) = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \eta\}. \tag{23}$$

If in addition $x_0 \neq 0, p(t_0) = S_1 x(t_0)$.

Proof. If $v_0(t)$ minimizes (19), then it also minimizes

$$\tilde{J}(v) \triangleq \frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^T \frac{1}{2}v^* R v \, dt - \lambda \int_{t_0}^T \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2}v^* Q_3 v\} \, dt. \tag{24}$$

By the maximum principle[11], there exists an adjoint response $\eta(t)$ such that the Hamiltonian

$$H(\eta, \zeta, v) = -\frac{1}{2}v^* R v + \lambda\{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2}v^* Q_3 v\} + \eta^*\{M\zeta + Nv\} \tag{25}$$

is maximized almost everywhere on $[t_0, T]$ by $v_0(t)$. Satisfaction of $\frac{\partial H}{\partial v} = 0$ yields

$$v_0(t) = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \eta\}. \tag{26}$$

The adjoint variable η satisfies

$$\frac{d\eta}{dt} = -\frac{\partial H}{\partial \zeta} = -M^* \eta - \lambda Q_1 \zeta - \lambda Q_2 v. \tag{27}$$

By the transversality conditions, we get the boundary conditions. □

Let

$$\tilde{M} = M + \lambda N (R - \lambda Q_3)^{-1} Q_2^*, \tag{28}$$

$$\tilde{N} = N (R - \lambda Q_3)^{-1} N^*, \tag{29}$$

and

$$\tilde{L} = -\lambda Q_1 - \lambda^2 Q_2 (R - \lambda Q_3)^{-1} Q_2^*. \tag{30}$$

Thus we have a two point boundary value problem given by

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \tilde{M} & \tilde{N} \\ \tilde{L} & -\tilde{M}^* \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}, \tag{31}$$

with

$$\begin{aligned} &x(t_0) = x_0, \ \psi(T) = 0, \\ &p(T) = 0, \ q(t_0) = 0, \\ &p(t_0) = S_1 x(t_0) \text{ if } x_0 \neq 0. \end{aligned} \tag{32}$$

We now give a criterion for the estimation of λ. Notice that $\lambda = \min_v \max_u J(u, v)$ and gives a measure of performance of the optimal controller under worst-case conditions corresponding to $v_0(t)$. In the H_∞ case, the evaluation of λ would entail the γ-iteration.

THEOREM 2. *Let λ be the smallest positive value for which the boundary value problem given by (31) and (32) has a solution (ζ, η) with $\int_{t_0}^{T} \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v\, dt > 0$, where $v \triangleq (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \eta\}$. Then λ is the minimum value of (19), (ζ, η) is an optimal pair and $v = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \eta\}$ is the worst exogenous input.*

Proof. It is clear from Theorem 1 that if $v_0(t)$ minimizes (19), then it satisfies (31) and (32), with λ being the minimum value of (19). Now suppose (ζ, η) satisfies (31) and (32) for some λ. Let $v = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \eta\}$. In the following equations $(\ , \ )$ denotes an inner product.

We have

$$\int_{t_0}^{T} ((R - \lambda Q_3)v, v)\, dt = \int_{t_0}^{T} (\lambda Q_2^* \zeta, v)\, dt + \int_{t_0}^{T} (N^* \eta, v)\, dt. \tag{33}$$

By equation (17), the second integral of (33) can be written as

$$\int_{t_0}^{T} (N^* \eta, v)\, dt = \int_{t_0}^{T} (\eta, Nv)\, dt = \int_{t_0}^{T} (\eta, \dot{\zeta} + M\zeta)\, dt. \tag{34}$$

An integration by parts and equations (20) and (29) yield

$$\int_{t_0}^{T} (\eta, \dot{\zeta} + M\zeta)\, dt = -x_0^* S_1 x_0 + \lambda \int_{t_0}^{T} (Q_1\zeta, \zeta)\, dt + \lambda \int_{t_0}^{T} (\zeta, Q_2 v)\, dt. \tag{35}$$

Substituting (35) in (33), we get

$$x_0^* S_1 x_0 + \int_{t_0}^{T} v^* R v\, dt = \lambda \int_{t_0}^{T} \{\zeta^* Q_1 \zeta + 2\zeta^* Q_2 v + v^* Q_3 v\}\, dt. \tag{36}$$

Thus, the cost associated with v is λ. Hence, if (ζ, η) is a nontrivial solution of the boundary value problem given by (31) and (32) for the smallest parameter $\lambda > 0$, then λ is the optimal value and (ζ, η) is an optimal pair. □

Note that the boundary value problem (31)-(32) has a solution with a nonvanishing denominator for (19) for at most a countably infinite values of λ. Theorem 2 gives a sufficient condition for an exogenous input to be optimal. Thus, Theorems 1 and 2 give a complete characterization of the worst-case exogenous input.

4. COMPUTATION OF λ

In this section, we consider the boundary value problem given by (31) and (32) assuming that $x(t_0) \neq 0$. Analogous theory can be developed in case $x_0 = 0$. Making use of the transition matrix, the solution of (31) can be expressed as

$$\begin{pmatrix} x(t) \\ \psi(t) \\ p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) & \phi_{13}(t,t_0) & \phi_{14}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) & \phi_{23}(t,t_0) & \phi_{24}(t,t_0) \\ \phi_{31}(t,t_0) & \phi_{32}(t,t_0) & \phi_{33}(t,t_0) & \phi_{34}(t,t_0) \\ \phi_{41}(t,t_0) & \phi_{42}(t,t_0) & \phi_{43}(t,t_0) & \phi_{44}(t,t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \\ p(t_0) \\ q(t_0) \end{pmatrix}. \tag{37}$$

The boundary conditions given by (32) yield

$$\begin{pmatrix} \phi_{21}(T,t_0) + \phi_{23}(T,t_0)S_1 & \phi_{22}(T,t_0) \\ \phi_{31}(T,t_0) + \phi_{33}(T,t_0)S_1 & \phi_{32}(T,t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \end{pmatrix} = 0. \tag{38}$$

Let

$$\tilde{\phi} = \begin{pmatrix} \phi_{21} + \phi_{23}S_1 & \phi_{22} \\ \phi_{31} + \phi_{33}S_1 & \phi_{32} \end{pmatrix}. \tag{39}$$

In view of (38) and (31)-(32), we have $\det(\tilde{\phi}(T,t_0)) = 0$ if and only if the solution (ζ,η) of (31)-(32) is not identically zero. Thus, we need the least positive λ which makes $\det(\tilde{\phi}(T,t_0)) = 0$ and the denominator of (19) positive. This can be obtained by doing a search with λ over an interval on which there is a change in the sign of the determinant.

We found the following algorithm to be numerically more stable since numbers of lesser magnitude are involved in the computation of the transition matrices in (38). We have

$$\begin{pmatrix} \zeta(T) \\ \eta(T) \end{pmatrix} = \phi(T, \frac{T+t_0}{2})\phi(\frac{T+t_0}{2}, t_0) \begin{pmatrix} \zeta(t_0) \\ \eta(t_0) \end{pmatrix}. \tag{38}$$

Let

$$\phi^{-1}(T, \frac{T+t_0}{2}) = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix}, \tag{41}$$

and

$$\phi(\frac{T+t_0}{2}, t_0) = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} & \nu_{14} \\ \nu_{21} & \nu_{22} & \nu_{23} & \nu_{24} \\ \nu_{31} & \nu_{32} & \nu_{33} & \nu_{34} \\ \nu_{41} & \nu_{42} & \nu_{43} & \nu_{44} \end{pmatrix}. \tag{42}$$

Making use of $p(t_0) = S_1 x(t_0), q(t_0) = \psi(T) = p(T) = 0$, we have

$$\begin{pmatrix} \xi_{11} & \xi_{14} \\ \xi_{21} & \xi_{24} \\ \xi_{31} & \xi_{34} \\ \xi_{41} & \xi_{44} \end{pmatrix} \begin{pmatrix} x(T) \\ q(T) \end{pmatrix} = \begin{pmatrix} \nu_{11} + \nu_{13}S_1 & \nu_{12} \\ \nu_{21} + \nu_{23}S_1 & \nu_{22} \\ \nu_{31} + \nu_{33}S_1 & \nu_{32} \\ \nu_{41} + \nu_{43}S_1 & \nu_{42} \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \end{pmatrix}. \tag{43}$$

The above equation has a nontrivial solution if and only if

$$\det \begin{pmatrix} \xi_{11} & \xi_{14} & \nu_{11} + \nu_{13}S_1 & \nu_{12} \\ \xi_{21} & \xi_{24} & \nu_{21} + \nu_{23}S_1 & \nu_{22} \\ \xi_{31} & \xi_{34} & \nu_{31} + \nu_{33}S_1 & \nu_{32} \\ \xi_{41} & \xi_{44} & \nu_{41} + \nu_{43}S_1 & \nu_{42} \end{pmatrix} = 0. \tag{44}$$

Thus, we need the least positive λ which makes the above determinant zero.

5. Solution in Terms of Riccati Equations

We now give the optimal solution in terms of solutions of two dynamic Riccati equations in the case where $x_0 \neq 0$.

THEOREM 3. *Let P be the solution of the initial value problem*

$$\dot{P} + P\tilde{M} + \tilde{M}^*P + P\tilde{N}P - \tilde{L} = 0, \tag{45}$$

$$P(t_0) = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{46}$$

Let

$$\theta = (R - \lambda Q_3)^{-1}(\lambda Q_2^* + N^*P) = (\theta_1 \quad \theta_2), \tag{47}$$

where θ_1 and θ_1 have equal number of columns. If $x(t_0) \neq 0$, let Z be the solution of the dynamic Riccati equation

$$\dot{Z} + Z(\hat{A} + G_1\theta_1) + (\hat{A}^* - G_2\theta_2)Z + Z(\hat{B} + G_1\theta_2)Z - \hat{C} - G_2\theta_1 = 0, \quad Z(T) = 0. \tag{48}$$

Then the worst-case exogenous input is given by

$$v = (\theta_1 + \theta_2 Z)x, \tag{49}$$

and the worst-case optimal controller is

$$u = W_3^{-1}((B_1^* - W_6\theta_2)Z - W_2^* - W_6\theta_1)x. \tag{50}$$

Proof. Letting $\eta = P\zeta$ in (31), we get (45). From (26) the worst-case exogenous input is $v = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* + N^*P)\zeta$. If $x(t_0) \neq 0$, letting $\psi = Zx$, and utilizing (47), we get $v = (\theta_1 + \theta_2 Z)x$. Also, equations (12) and (13) yield (48). From (10), we get the worst-case optimal controller given by (50). □

Note that the worst-case optimal control given by (50) has time-varying feedback gain even when the linear system is time-invariant.

6. PERFORMANCE ROBUSTNESS

In this section we develop a formula for the variation of λ when there are parameter variations in the system matrices. Note that this expression for the variation in λ takes into account the corresponding variations in the optimal controller and the worst-exogenous input. Usually when a controller is synthesized with respect to the nominal values of system matrices, its gains do not change with parameter variations. Hence, later on in this section we will derive an expression for the variation in λ assuming that there is no variation of the optimal gain matrix.

For this consider (1) and (2). Let μ denote the variation in λ for elemental variations $\delta A, \delta B_1, \delta B_2, \delta C, \delta D$, and δE in the matrices A, B_1, B_2, C, D and E. From equations (31) and (32) of Section 3, we have the following boundary value problem given by

$$\dot{\zeta} = \tilde{M}\zeta + \tilde{N}\eta, \tag{51}$$

$$\dot{\eta} = \tilde{L}\zeta - \tilde{M}^*\eta, \tag{52}$$

with

$$\begin{aligned} &x(t_0) = x_0, \psi(T) = 0, \\ &p(T) = 0, q(t_0) = 0, \\ &p(t_0) = S_1 x(t_0) \text{ if } x_0 \neq 0. \end{aligned} \tag{53}$$

Let $\delta\tilde{M}, \delta\tilde{N}$, and $\delta\tilde{L}$ be the variations in $\tilde{M}, \tilde{N}$, and $\tilde{L}$ owing to the variations δA, δB_1, δB_2, δC, δD, and δE. Let the corresponding variation in λ be denoted by μ.

Let $\Lambda = (R - \lambda Q_3)^{-1}$. From (28)-(30), we get

$$\delta\tilde{M} = I_1 + \mu I_2, \tag{54}$$

$$\delta\tilde{N} = J_1 + \mu J_2, \tag{55}$$

$$\delta\tilde{L} = K_1 + \mu K_2, \tag{56}$$

where

$$I_1 = \delta M + \lambda N\Lambda\delta Q_2^* + \lambda\delta N\Lambda Q_2^* + \lambda^2 N\Lambda\delta Q_3\Lambda Q_2^*, \tag{57}$$

$$I_2 = N\Lambda Q_2^* + \lambda N\Lambda Q_3\Lambda Q_3^*, \tag{58}$$

$$J_1 = N\Lambda\delta N^* + \delta N\Lambda N^* + \lambda N\Lambda\delta Q_3\Lambda N^*, \tag{59}$$

$$J_2 = N\Lambda Q_3\Lambda N^*, \tag{60}$$

$$K_1 = -\lambda\delta Q_1 - \lambda^2\{\delta Q_2\Lambda Q_2^* + Q_2\Lambda\delta Q_2^*\} - \lambda^3 Q_2\Lambda\delta Q_3\Lambda Q_2^*, \tag{61}$$

$$K_2 = -Q_1 - 2\lambda Q_2\Lambda Q_2^* - \lambda^2 Q_2\Lambda Q_3\Lambda Q_2^* \tag{62}$$

For the sake of simplicity, we now derive an expression for μ in terms of the variations $\delta M, \delta N, \delta Q_1, \delta Q_2$, and δQ_3.

Let ζ_1 and η_1 represent variations in ζ and η owing to $\delta M, \delta N, \delta Q_1, \delta Q_2$, and δQ_3. We have the following set of equations that are satisfied by ζ_1 and η_1:

$$\dot{\zeta}_1 = \tilde{M}\zeta_1 + \tilde{N}\eta_1 + (I_1 + \mu I_2)\zeta + (J_1 + \mu J_2)\eta, \tag{63}$$

$$\dot{\eta}_1 = \tilde{L}\zeta_1 - \tilde{M}^*\eta_1 + (K_1 + \mu K_2)\zeta - (I_1 + \mu I_2)^*\eta, \tag{64}$$

with

$$\begin{aligned} &x_1(t_0) = x_{10}, \psi_1(T) = 0, \\ &p_1(T) = 0, q_1(t_0) = 0, \\ &p_1(t_0) = S_1 x_{10} \text{ if } x_0 \neq 0. \end{aligned} \tag{65}$$

Note that the subscript 1 of a variable in (65) denotes the corresponding variation of that variable.

THEOREM 4. *Let $v \triangleq \Lambda\{\lambda Q_2^*\zeta + N^*\eta.\}$ Then the variation μ in performance is given by*

$$\mu = \frac{\int_{t_0}^T \{\zeta^* K_1\zeta - 2\zeta^* I_2^*\eta - \eta^* J_1\eta\}\, dt}{\int_{t_0}^T \{\zeta^* Q_1\zeta + 2\zeta^* Q_2 v + v^* Q_3 v\}\, dt}. \tag{66}$$

Proof. From (64), we get

$$\int_{t_0}^T \zeta^*\dot{\eta}_1\, dt = \int_{t_0}^T \{\zeta^*\tilde{L}\zeta_1 - \zeta^*\tilde{M}^*\eta_1 + \zeta^*(K_1 + \mu K_2)\zeta - \zeta^*(I_1 + \mu I_2)^*\eta\}\, dt. \tag{67}$$

Integrating the left side of (67) by parts and making use of (51), (53) and (65), we get

$$\begin{aligned} -x_0^* S_1 x_{10} - \int_{t_0}^T \eta^*\tilde{N}\eta_1\, dt &= \int_{t_0}^T \zeta^*\tilde{L}\zeta_1\, dt \\ &+ \int_{t_0}^T \zeta^*(K_1 + \mu K_2)\zeta\, dt - \int_{t_0}^T \zeta^*(I_1 + \mu I_2)^*\eta\, dt. \end{aligned} \tag{68}$$

By (52), the first integral on the right side of (58) is written as

$$\int_{t_0}^{T} \zeta^* \tilde{L} \zeta_1 \, dt = \int_{t_0}^{T} (\dot{\eta} + \tilde{M}^* \eta)^* \zeta_1 \, dt. \tag{69}$$

An integration by parts and equations (53), (63) and (65) yield

$$\int_{t_0}^{T} \zeta^* \tilde{L} \zeta_1 \, dt = -x_0^* S_1 x_{10} - \int_{t_0}^{T} \eta^* \tilde{N} \eta_1 \, dt - \int_{t_0}^{T} \eta^* (I_1 + \mu I_2) \zeta \, dt - \int_{t_0}^{T} \eta^* (J_1 + \mu J_2) \eta \, dt. \tag{70}$$

Substituting (70) in (68) and simplifying, we get

$$\mu = \frac{\int_{t_0}^{T} \{\zeta^* K_1 \zeta - 2\zeta^* I_1^* \eta - \eta^* J_1 \eta\} \, dt}{\int_{t_0}^{T} \{\eta^* J_2 \eta + 2\eta^* I_2 \zeta - \zeta^* K_2 \zeta\} \, dt}. \tag{71}$$

A little algebra shows that the denominator of (71) equals the denominator of (66) □

Since μ given by (66) is linear in the elemental variations $\delta A, \delta B_1, \delta B_2, \delta C, \delta D$ and δE, at least in the time-invariant case the worst degradation in performance can be easily obtained once the range of uncertainty of the parameters is known.

Now we consider the case where $x_0 \neq 0$. Assume that the state feedback controller is determined by the nominal system matrices and is fixed. We derive a formula for the variation of λ under these conditions. Since λ gives a measure of performance of the optimal controller under worst-case conditions, we can get an idea of the degradation in performance owing to parameter variations. Equation (49) is written as

$$u = W_3^{-1} \big((B_1^* - W_6 \theta_2) Z - W_2^* - W_6 \theta_1 \big) x = K(t) x, \tag{72}$$

where $K(t)$ is now fixed. Let $\tilde{A} = A + B_1 K$. Equation (1) can be written as

$$\dot{x} = \tilde{A}(t) + B_2(t) v, \tag{73}$$

with v chosen to minimize

$$\frac{\frac{1}{2} x_0^* S_1 x_0 + \int_{t_0}^{T} \frac{1}{2} v^* R v \, dt}{\int_{t_0}^{T} \{\frac{1}{2} x^* \tilde{W}_1 x + x^* \tilde{W}_2 v + \frac{1}{2} v^* \tilde{W}_3 v\} \, dt}. \tag{74}$$

Note that

$$\lambda = \min_v \frac{\frac{1}{2}x_0^* S_1 x_0 + \int_{t_0}^T \frac{1}{2} v^* R v \, dt}{\int_{t_0}^T \{\frac{1}{2} x^* \tilde{W}_1 x + x^* \tilde{W}_2 v + \frac{1}{2} v^* \tilde{W}_3 v\} \, dt}. \tag{75}$$

Let $\Omega = (R - \lambda \tilde{W}_3)^{-1}$ and

$$\hat{M} = \tilde{A} + \lambda B_2 \Omega \tilde{W}_2^*, \tag{76}$$

$$\hat{N} = B_2 \Omega B_2^*, \tag{77}$$

$$\hat{L} = -\lambda \tilde{W}_1 - \lambda^2 \tilde{W}_2 \Omega \tilde{W}_2^*. \tag{78}$$

The above minimization problem yields the two-point boundary value problem

$$\begin{pmatrix} \dot{x} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} \hat{M} & \hat{N} \\ \hat{L} & -\hat{M}^* \end{pmatrix} \begin{pmatrix} x \\ \beta \end{pmatrix}, \tag{79}$$

$$x(t_0) = 0, \beta(t_0) = S_1 x_0, \beta(T) = 0, \tag{80}$$

where β is the adjoint variable and the worst exogenous input $v = \Omega(B_2^* \beta + \lambda \tilde{W}_2^* x)$.

Let $\delta\tilde{A}, \delta B_2, \delta\tilde{W}_1, \delta\tilde{W}_2$, and $\delta\tilde{W}_3$ be the variations in $\tilde{A}, B_2, \tilde{W}_1, \tilde{W}_2$, and $\tilde{W}_3$ corresponding to $\delta A, \delta B_1, \delta B_2, \delta C, \delta D$, and δE. Note that since $K(t)$ is fixed, $\delta\tilde{A} = \delta A + \delta B_1 K$. Let the variation in λ be now denoted by $\tilde{\mu}$. Utilizing a similar analysis as in the derivation of (66), we can get

$$\tilde{\mu} = \frac{\int_{t_0}^T \{x^* \hat{K}_1 x - 2x^* \hat{I}_1^* \beta - \beta^* \hat{J}_1 \beta\} \, dt}{\int_{t_0}^T \{x^* \tilde{W}_1 x + 2x^* \tilde{W}_2 v + v^* \tilde{W}_3 v\} \, dt}, \tag{81}$$

where

$$\hat{I}_1 = \delta\tilde{A} + \lambda\{B_2 \Omega \delta\tilde{W}_2^* + \delta B_2 \Omega \tilde{W}_2^*\} + \lambda^2 B_2 \Omega \delta\tilde{W}_3 \Omega \tilde{W}_2^*, \tag{82}$$

$$\hat{J}_1 = B_2 \Omega \delta B_2^* + \delta B_2 \Omega B_2^* + \lambda B_2 \Omega \delta\tilde{W}_3 \Omega B_2^*, \tag{83}$$

$$\hat{K}_1 = -\lambda \delta\tilde{W}_1 - \lambda^2 \{\delta\tilde{W}_2 \Omega \tilde{W}_2^* + \tilde{W}_2 \Omega \delta\tilde{W}_2^*\} - \lambda^3 \tilde{W}_2 \Omega \delta\tilde{W}_3 \Omega \tilde{W}_2^*. \tag{84}$$

Since $\tilde{\mu}$ is linear in the variations, the worst degradation in the performance of the optimal controller can be easily computed in the time-invariant case. The worst value of $\tilde{\mu}$ gives an idea of the measure of performance robustness of the optimal controller.

7. An Example

In order to illustrate the basic theory, we will work out a simple example. The system is described by the equation

$$\dot{x} = -x + u + v, \qquad x(0) = x_0 \neq 0, \tag{85}$$

and the objective is to choose u and v such that

$$\min_v \max_u \frac{\frac{1}{2}x_0^2 + \int_0^1 \frac{1}{2}v^2\,dt}{\int_0^1 \frac{1}{2}(x^2+u^2+v^2)\,dt} \tag{86}$$

is attained.

First of all, minimizing $\int_0^1 \frac{1}{2}(x^2+u^2+v^2)\,dt$ over $u(t)$ for a given $v(t)$, we get

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \tag{87}$$

$$u = \psi, \tag{88}$$

$$x(0) = x_0, \psi(1) = 0, \tag{89}$$

where ψ is the adjoint variable. Now we need to choose v to minimize

$$\frac{\frac{1}{2}x_0^2 + \int_0^1 \frac{1}{2}v^2\,dt}{\int_0^1 \frac{1}{2}(x^2+\psi^2+v^2)\,dt}. \tag{90}$$

Let λ be the minimum value of (90). Denoting the adjoint variables associated with x and ψ by η^1 and η^2 respectively, we get

$$\frac{d\eta^1}{dt} = -\lambda x + \eta^1 - \eta^2, \tag{91}$$

$$\frac{d\eta^2}{dt} = -\lambda\psi - \eta^1 - \eta^2, \tag{92}$$

$$\eta^1(0) = x_0, \eta^2(0) = 0, \eta^1(1) = 0, \tag{93}$$

$$v = \frac{\eta^1}{1-\lambda}. \tag{94}$$

Thus, we have

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \\ \dot{\eta}^1 \\ \dot{\eta}^2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & \frac{1}{1-\lambda} & 0 \\ 1 & 1 & 0 & 0 \\ -\lambda & 0 & 1 & -1 \\ 0 & -\lambda & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ \psi \\ \eta^1 \\ \eta^2 \end{pmatrix}, \tag{95}$$

$$x(0) = x_0, \psi(1) = 0, \eta^1(0) = x_0, \eta^2(0) = 0, \eta^1(1) = 0. \tag{96}$$

According to the theory of Section 3, λ is the least positive value for which the boundary value problem (95)-(96) has a nonzero solution.

Let ϕ be the transition matrix of the system given by (95) at $t = 1$. Solving (95) and employing the boundary conditions at $t = 1$, we get

$$0 = \begin{pmatrix} \psi(1) \\ \eta^1(1) \end{pmatrix} = F(\lambda) \begin{pmatrix} x(0) \\ \psi(0) \end{pmatrix}, \tag{97}$$

where

$$F(\lambda) = \begin{pmatrix} \phi_{21} + \phi_{23} & \phi_{22} \\ \phi_{31} + \phi_{33} & \phi_{32} \end{pmatrix}. \tag{98}$$

Thus, we need the first positive λ which makes $\det(F(\lambda)) = 0$. This value of λ is 0.8276. It can be easily shown that with the initial condition $x(0) = 0$, the value of λ would have been 0.85947. This of course is the first positive λ which makes

$$\det \begin{pmatrix} \phi_{22} & \phi_{23} \\ \phi_{32} & \phi_{33} \end{pmatrix} = 0. \tag{99}$$

Now the Riccati equations in Theorem 3 can be easily solved to obtain the worst-case optimal controller and the worst-case exogenous input.

8. CONCLUSIONS

In this chapter we presented a solution to the finite interval worst-case state feedback controller in terms of solutions of two dynamic Riccati equations. These equations are easy to solve since only one of the two equations is dependent on the solution of the other. the procedure yields optimal solutions instead of suboptimal ones normally obtained by H_∞ methods. Also, an expression is derived for the degradation in performance of the optimal controller in terms of parameter variations.

IV. References

[1] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, *State-space solutions to standard H_2 and H_∞ control problems*, IEEE Transactions on Automatic Control, Vol. 34, 1989, pp. 831-847.

[2] P. P. Khargonekar, *State-space H_∞ Control Theory*, Control Group No. CGR-27, College of Engineering, The University of Michigan, 1989, to appear in "Mathematical System Theory: The Influence of R. E. Kalman," edited by A. C. Antoulas, Springer-Verlag.

[3] P. P. Khargonekar, K. M. Nagpal, and K. R. Poolla, *H_∞ control with transients*, SIAM Journal on Control and Optimization, to appear.

[4] R. Ravi, K. M. Nagpal, and P. P. Khargonekar, *H^∞ control of linear time-varying systems: A state space approach*, SIAM Journal on Control and Optimization, to appear.

[5] G. Tadmor, *Worst-case design in the time domain: the maximum principle and the standard H_∞ problem*, Mathematics of Control, Signals, and Systems, Vol. 3, 1990, pp. 301-324.

[6] M. B. Subrahmanyam, *Optimal Control with a Worst-Case Performance Criterion and Applications*, Lecture Notes in Control and Information Sciences, No. 145, Springer-Verlag, New York, 1990.

[7] ———, *Optimal disturbance rejection and performance robustness in linear systems*, Proc. 33rd Midwest Symposium on Circuits and Systems, August 1990; Journal of Mathematical Analysis and Applications, to appear.

[8] ———, *Synthesis of finite-interval H_∞ controllers by state-space methods*, Journal of Guidance, Control, and Dynamics, Vol. 13, 1990, pp. 624-629.

[9] ———————, Worst-case performance measures for linear control problems, Proc. 1990 IEEE Conference on Decision and Control, pp. 2439-2443.

[10] M. B. SUBRAHMANYAM AND M. STEINBERG, *Model reduction with a finite-interval H_∞ criterion*, Proc. AIAA Guidance, Navigation and Control Conference, 1990, pp. 1419-1427.

[11] L. S. PONTRYAGIN, V. G. BOLTYANSKI, R. V. GAMKRELIDZE, AND E. F. MISCHENKO, *The Mathematical Theory of Optimal Processes*, Interscience Publishers, New York, 1962.

DISTRIBUTION LIST
Report No. NADC-91005-60

	No. of Copies
Defense Technical Information Center ATTN: DTIC-FDAB Cameron Station BG5 Alexandria, VA 22304-6145	2
Center for Naval Analyses 4401 Fort Avenue P.O. Box 16268 Alexandria, VA 22302-0268	1
Naval Air Development Center Warminster, PA 18974-5000 (2 for Code 8131) (200 for Code 6012 - B. Subrahmanyam)	202
Office of Naval Research 800 N. Quincy St. Arlington, VA 22217-5000 Attn: Dr. Jack Hansen (ONR-121)	1
Naval Postgraduate School Monterey, CA 93940 Attn: Dr. D.J. Collins (Dept. of Aeronautics)	1
Office of Naval Technology Arlington, VA 22217-5000	1